



**CONSIGLIO NAZIONALE
DELLE RICERCHE**

Goffredo Haus [^], Paolo Morini [^]

**TEMPER: A System for Music Synthesis
from Animated Tessellations**

N. 7/90

8/92

**PROGETTO FINALIZZATO
SISTEMI INFORMATICI E CALCOLO PARALLELO**

SOTTOPROGETTO 7

SISTEMI DI SUPPORTO AL LAVORO INTELLETTUALE

**PROGETTO FINALIZZATO
SISTEMI INFORMATICI E CALCOLO PARALLELO**

SOTTOPROGETTO 7
Sistemi di Supporto al Lavoro Intellettuale
Coordinatore Cesare Maioli

Goffredo Haus [^], Paolo Morini [^]

**TEMPER: A System for Music Synthesis
from Animated Tessellations**

N. 7/90

8/92

Reprint

[^] LIM - Laboratorio di Informatica Musicale
Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano.

TEMPER: A System for Music Synthesis from Animated Tessellations

Goffredo Haus and
Paolo Morini

THE QUESTION OF PAINTING AND MUSIC

Much literature has been written and many works have been completed in the area of audio-visual composition and performance. Most of these works start with the implicit assumption that sound and images can run concurrently without any kind of interaction and correlation. Usually only the synchronization of elements in time—sound linked with video or film—is attempted. However, there are a number of works that, while not specifically devoted to audio-visual objectives, have given more importance to correlating graphic and musical information, such as work involving the graphic representation of musical information, either to define notational systems [1] or to automatically produce traditional [2] or auditory [3] scores. In this article, we use the term 'traditional score' when referring to a common music notation score; and the term 'auditory score' when referring to a kind of music notation used to help listeners understand musical sequences and structures—particularly when the music either cannot be or is not represented by means of a traditional score (e.g. electronic, computer, ethnic musics).

Some of Kandinsky's written works are illuminating with respect to analogies between graphics and music [4]. Several articles published in *Leonardo* provide rich overviews of work in music visualization [5]. The stimulating work of Whitney [6] involves the synthesis of audio-visual processes, with a high correlation between auditory and visual aesthetic information. The work of Brian Evans is also significant in this field [7]. In addition, paintings have inspired the structure of music in many works: in the work of Cervetti, paintings by Bosch are depicted in music [8]; in the work of Brecker, the structure of a painting by M. C. Escher determines the process with which one rhythmic pattern gradually transforms into a second one [9]. Yet other works are based on fractals as the generative structures both for graphics and music.

Our research can be considered within the framework of the works we have mentioned, but our aim is to go beyond the traditional; we wish to investigate several kinds of aesthetic expressions—literature, painting, music, architecture and other forms of art—in order to find out if common structures exist among them. Our ultimate goal is the definition of a knowledge base suitable for the description of

multimedia compositions and performances in which there is a high level of correlation among different media.

We conducted a series of experimental works devoted to building our knowledge base. Some of our experiences bear mentioning: in one work, the metric-rhythmic structure underlying the dialog between Othello and Iago in Shakespeare's *Othello* was translated into music; another project involved the development of the computer program *Traslitteratore* [10], which allows the translation of any literary text (i.e. any text file) into a musical composition that respects metrics and accentuations of natural language. Other projects are in progress: for example, data received from the galaxy UGC 9567 have been translated into sound by means of temporal compression and the application of suitable scale factors for amplitude and frequency values [11]. All of this work concerns correlations between graphic animation and musical processes.

Within the framework of our general project for the definition of a knowledge base, we are concentrating on flows and limits in the process of aesthetic perception. A central point in the design of new kinds of aesthetic works is certainly the definition of suitable audio-visual 'stews'. Viewers must not be bored by information flows that are either too poor or too repetitive (which is the same), or overwhelmed by too rich or too varied an informative flow [12]. Our working hypothesis is that the whole audience can enjoy the audio-visual work only if the quality and quantity is within a certain range, which may vary from individual to individual but cannot exceed certain limits. Note that even

ABSTRACT

The authors discuss the results of research carried out since 1987 at Laboratorio di Informatica Musicale (LIM) involving the association of graphics and music, with special attention to drawings based on tessellations. After a theoretical investigation primarily devoted to the creation of a conceptual framework, the authors developed a prototypal computer program, TEMPER. This program allows for the editing of tessellations and for the definition of performance paths that include the animation of tessellations and the synthesis of music pieces obtained by translating graphic animations.

Goffredo Haus (researcher). Laboratorio di Informatica Musicale, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, via Comelico, 39, I-20135 Milan, Italy.

Paolo Morini (researcher). Laboratorio di Informatica Musicale, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, via Comelico, 39, I-20135 Milan, Italy.

Received 9 January 1991.

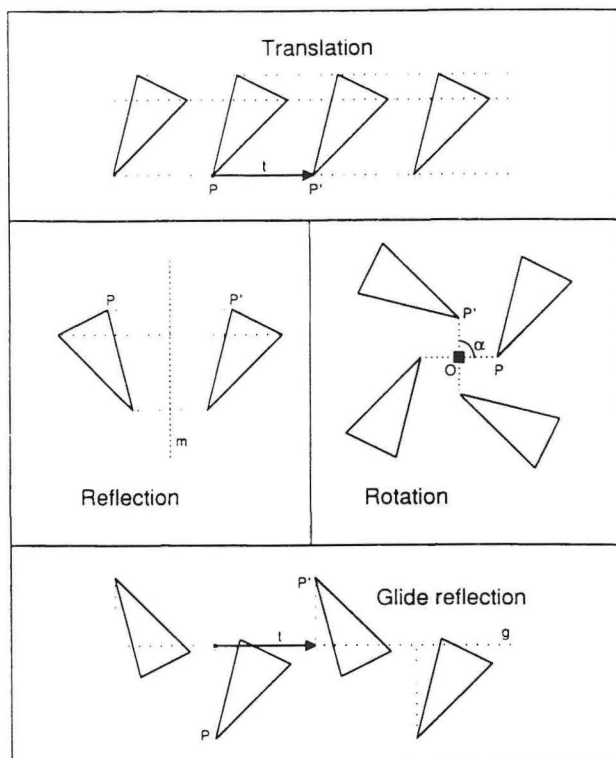


Fig. 1. Examples of the four symmetry operations allowed in the plane. Each point in the plane is transformed into another (for example P into P') by the symmetry operations; these are outlined by the vector t for the translation, the mirror line m for the reflection, the angle α for the rotation and the mirror line g together with the vector t for the glide reflection. The illustrations for the translation and glide reflection operations should be considered as infinitely and repeatedly extended in the direction of the vector t .

when the quantity of information given to members of an audience is equal, receiving capabilities may differ, depending on qualities of the information itself. A trivial but effective example would be that the same phrase, expressed in different natural languages, is of quite different value depending on the listener's mother-tongue. In other words, we think that the design of audio-visual works may be regarded as a kind of qualitative and quantitative 'information-flow engineering'.

CORRELATIONS BETWEEN GRAPHICS AND MUSIC

We are interested in a special kind of graphic process known as *tessellations*, or tilings—drawings made up of one or more shapes that fill the plane without gaps between them when appropriate symmetry operations such as rotations, reflections and translations are applied to them. Drawings of this kind are well known in the works of Dutch artist Escher [13–15].

From a certain point of view, tessellations show surprising affinity to musical pieces; in particular, graphic symmetry operations are closely related to common musical operations: there are mirror-like musical pieces that are similar to graphic reflections, tonality transposition and time repetitions in music that are similar to translation, cyclic tonality changes that recall rotations, and so on [16].

We were interested in checking to see if these affinities

are only superficial or if they have deeper roots. If the latter, we also wanted to discover if it is possible to associate a tessellation to a musical piece. If so, then we wanted to know if this association would correlate the artistic contents of the graphics with the music; last, we wished to investigate whether this process can be enacted by a computer program.

In dealing with these problems, we intended to obtain a work that presented a solid theoretical base as well as practical results. This requirement brought some additional difficulties, so that some ideas were temporarily shelved because of implementation problems. On the other hand we discovered things not strictly linked to theory. We accompanied these studies with a program; our aim was not only to obtain a practical demonstration of the theory, but also to develop a graphic and musical instrument that can actually be utilized for audio-visual performances [17].

Gathering together practical and theoretical requirements, we mainly followed these development guidelines:

1. The association between graphic and musical elements should be attempted only from graphics to music. The reverse, or a one-to-one correspondence, is possible, but could not easily be done in a proper manner because of quantizing problems.
2. Graphic-to-music transformation must be done according to an exact function, whose behaviour can be modified by the 'composer' through special parameters. There should not exist any casualness in the transformation; neither should it be possible to work on the music independently from the graphics.
3. The association should be easily perceived by the audio-visual observer; therefore it is important to note the limitations of the human brain relative to the capabilities of perception, abstraction and concentration.
4. The software should be able to perform the graphics and music in real time; only in this manner is it possible to obtain results that can be utilized for a demonstrative or artistic purpose.

Tessellations

The graphic side of our work consists of the study of tessellations. In particular we considered a kind of tessellation called *isohedral*. A tessellation can be obtained through the partition of an unlimited plane in sets of limited and connected points, so that every point in the plane belongs to one and only one set. From a graphic point of view every set is a shape; if all the shapes on the plane are directly or reflectively congruent—that is, they can be superimposed by an isometry—the tessellation is said to be monohedral; if, on applying the isometry needed to superimpose any two shapes, the whole tessellation superimposes on itself, then the tessellation is isohedral.

For the study of tessellations, we used the crystallographic theory of symmetry groups [18]. The details can be found in several publications, but we will point out here that in accordance with this theory we start from the concept of isometry [19], the operation that relates two congruent objects in the space. An isometry is called a symmetry operation if it can be applied to the whole space instead of to a single object, that is, if the spaces before and after the application of the isometry are identical.

There are only four kinds of symmetry operations in the plane: translations, reflections, rotations and glide reflections (Fig. 1). The combination of two symmetry operations is another symmetry operation; so interaction between a set

of operations generates other operations—however, there are limits on the kinds of operations possible. For tessellations, it is compulsory to have two non-parallel translation operations because otherwise it would not be possible to fill up the plane.

As a result of these considerations, there are 17 allowed combinations of symmetry operations. It is possible to demonstrate that every set of allowed symmetry operations, joined to the binary operation of composition (that yields another symmetry operation), forms a group (in the mathematical sense of the word). For this reason these combinations are named symmetry groups [20]. The ones in which we are interested are called bidimensional (with exactly two translations) and monochromatic (no symmetry operations will change the shapes' colour).

This leads to an important demonstration: to any isohedral tessellation we can apply all the symmetry operations in one of the 17 bidimensional monochromatic spatial groups. We say that the tessellation belongs to that symmetry group [21,22].

To define a tessellation we first need a shape and a group of symmetry operations. But we have some more points to develop. It is obvious that a generic shape cannot belong to any group; hence it is not easy to build a tessellation starting from an isolated shape. Besides, every attempt to modify a shape to fit to the adjacent one will also modify all the other shapes, creating holes that are difficult to fill.

Furthermore, consider the fact that the border of a shape is also part of the border of the adjacent one; we call the common border between two adjacent shapes a *side*, and we call the ends of sides *vertices*. Sides join only in vertices, and three or more sides necessarily join in a vertex. It is easy to find out that two tessellations belonging to the same symmetry group can be topologically different—that is, we cannot stretch one into the other. There are 81 different ways to build topologically distinct tessellations, as is shown in a study by Grünbaum and Shephard [23].

We looked for and found a method, suitable to be implemented on a computer, to facilitate the building of tessellations. We start from a set of 81 models of tessellations, supplying a single shape that is a simple example for every scheme; then we allow the sides of this shape to stretch. In fact, it is enough to stretch only some parts of the sides, because the other parts are obtained by transpositions: every stretch of a part should be reproduced on all the parts of the sides that can be reached through the application of a symmetry operation. Because this mechanism does not modify the symmetry operations, the shapes so obtained necessarily belong to the same tessellation schema and can then tile the plane.

The information needed to main-

tain this kind of formulation belongs to five hierarchically ordered categories:

Symmetry group: one of the 17 bidimensional monochromatic spatial groups.

Tessellation scheme: one of the 81 isohedral tessellation schemes, chosen in accordance with the symmetry group.

Translation vectors: the two vectors corresponding to the shortest symmetry operation of non-parallel translation, defining the scale and the orientation of the tessellation. Some symmetry groups impose bounds on the ratio length of the vectors and on the angle subtended by them.

Vertices: the joining points of sides of adjacent shapes; the symmetry group and translation vectors impose bounds on the vertices' reciprocal position.

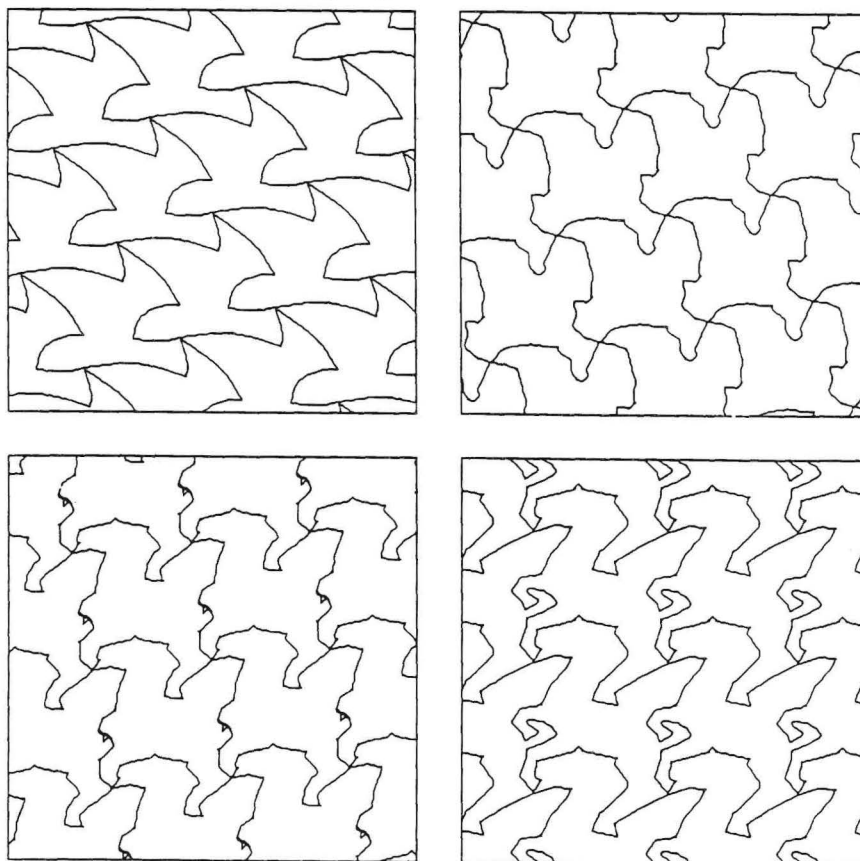
Sides: the lines defining a path between the adjacent vertices of a shape. Some parts are defined by a symmetry operation applied on another part; the independent parts are free from bounds, even if in special cases crosses and coincidences of sides cause the main shape to break into several.

This method proved to be particularly suitable to build tessellations; the 'weft' of the symmetry operation is implicit (in a computer program it is automatic); therefore it is possible to concentrate on the shape to be built without wasting attention on its insertion into the whole drawing.

The Graphics and Music Connection

The main problem was how to associate graphic and musical elements. To settle the question with a correct theoretical approach and at the same time with an association that is

Fig. 2. A three-step transformation from birds to winged horses (shapes are inspired by Escher's works) performed by the TEMPER program. Notice that lines that appear curved are in fact composed by short straight segments.



not too hidden to an observer-listener, it was necessary to begin a study of the nature of graphics observation and music listening.

The first phase of this study was an analysis of the dimensions defining graphic and musical spaces; the decomposition into dimensions can be made in several ways, but we thought that every attempt should fall within one of the following ways:

Spatial: the events to be examined are simple elements (i.e. points and segments in graphics; notes in music); these elements have to be lying on a coordinate system such as a bidimensional plane in graphics or time and pitch in music.

Functional: the events to be examined are complex elements on which compositional and transformational functions can operate; examples of this kind of function could be symmetry operators in graphics, or transposition and mirroring in music.

After an opportune choice for the decomposition of the two representational spaces is made, it is necessary to establish pairs between the chosen dimensions, so that a given event in a space can be translated into a corresponding event in the other space only by transforming its coordinates. The choice of dimensions and the coupling of them is without doubt the most critical phase of the whole work because it defines the transformation between graphics and music; so the results strictly depend on these choices.

At this point we met a serious problem: how to treat the temporal dimension. In fact this dimension is usually absent in graphic work (except for subjective aspects); while it is a fundamental component in music. This difference is the main cause of the dissimilar kinds of mental images that form in our minds in the two cases. Since one of our aims is to make as evident as possible the association between graphics and music, a careful examination of the temporal question is needed.

To properly treat the temporal dimension, it is necessary to balance the quantity of information contained in the graphics and music, relative to the capabilities of the human brain receiving this information, and to compare it with previously received information. This balancing operation is difficult and delicate because reception capabilities [24] and short-term associative memory have dissimilar characteristics for visual and auditory information.

In accordance with the work's requirements, and bearing in mind the problems previously illustrated, we had to make some decisions about which methods to use for the association of graphics and music.

In deciding between the spatial or functional approach, we chose the former for several reasons:

1. The functional approach is surely more significant; indeed it is related to the spatial one as semantics is to syntax; nevertheless this advantage works against itself for the more complex and less intuitive association.
2. Operations work on composite entities that cannot be directly associated with a single musical event; therefore a second level of interpretation is necessary for every entity. For this step the spatial approach becomes indispensable.
3. A translation made with the spatial approach will implicitly reflect the behaviour of the operations that define the functional approach; without recognizing the underlying structure, every element of original and transformed entities is translated in the same way.

For these reasons the results of a spatial approach are simpler than the results of a functional approach, without a notable loss of significance.

Following the choice of adopting the spatial approach it was compulsory to use horizontal and vertical dimensions to define the drawing plane, and to associate these dimensions with tempo and pitches, respectively. The result is a scan of the plane from left to right; graphic elements are translated into musical ones as they are touched by the scan; the higher the graphic element in the drawing, the higher the pitch of the note generated.

This solution could seem oversimplified, but the pairing of dimensions obtained a good balance of graphic and musical psychoperceptive aspects. Indeed we notice that an observer of a graphic work usually considers it more natural to find the more solid objects in the lower part of a drawing, because it creates a feeling of gravity and, from a musical point of view, low-pitched notes give a heavier feeling.

Another important consideration was that horizontal time scanning could be made from right to left, although, for Western people, reading from left to right (including musical staves) is a more natural motion [25–27].

We have yet to define the graphic elements necessary to be translated. In accordance with the spatial approach, these elements are the strokes of the drawing, that is, the sides of the tessellating shapes. But if we notice that music is commonly perceived as a sequence of discrete events (notes), not as a continuously varying sound [28], a better choice is to consider the sides made up only of broken lines; then their vertices can easily be associated with single discrete musical events. Therefore a curved line should be approximated by a sequence of straight segments.

All these choices express the metaphor of the melodic line (a graphic word—line—for a sequence of notes) in a way that calls to mind the musical staff. This means that the interpretation of the music generated by tessellations is not dissimilar to that used by musicians performing a piece, so the brain can apply a well-stated mechanism.

In addition, we considered it to be of great importance to define the musical scale used during translation, that is, which pitch values correspond to the scale degrees, or notes. It is useful to note that for music, once the key or tonality with which one is playing is fixed, every degree does not have the same importance. We can roughly set two groups: pitches inside and outside of the scale. The latter are generally used only as intermediate notes, because otherwise they could easily result in unpleasant dissonance. To further justify this choice we note that there are instruments, especially the wind instruments, that are tuned in a particular tonality, and that even common music notation makes a distinction between pitches inside the scale and outside the scale, specifying the second pitches with special symbols like sharps and flats.

We have yet to solve the problem of the temporal dimension: left-to-right scanning is a valid choice, but it is usually completed in too short a time to provide a feeling of completion. Rather, the corresponding graphics are generally perceived as the finished work. This causes a lack of balance in informational content.

To eliminate this problem we decided not to use a fixed image, but, rather, to gradually transform one tessellation into another one with a sequence of intermediate tessellations. In this way the graphics are supplied with a temporal dimension comparable to the musical one, since a still image

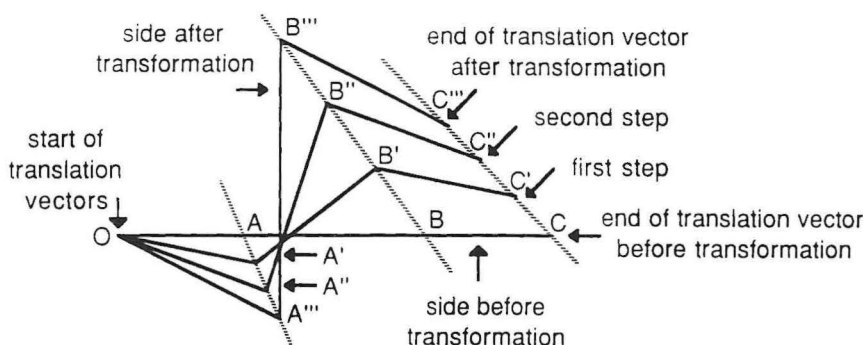


Fig. 3. The transformation function: a three-step transformation from a straight line to a three-segment broken line. The relative length of the segment OA with respect to the total length of the broken line OC is the same as the length of the segments OA', OA'' and OA''' with respect to the lines OC', OC'' and OC''', respectively. The same relationship applies to all the corresponding segments within the broken lines.

is changed into a slow animation, whose steps are scanned from left to right at a constant speed. The contemporaneous performance of sequences of tessellations and of the corresponding sequences of notes result in a greater feeling of completion and satisfaction.

For this purpose we studied a special transformation function that acts on the sides (broken lines) of the tessellations. This function is able to continuously generate tessellations between any two tessellations; therefore it is possible to execute a gradual transformation (with the required number of intermediate steps) from one tessellation to a second one, from the second to a third, and so on (at this point the function applies only to tessellations of the same tessellation scheme). The quality of the transformation function was also evaluated in an experimental way, by observing the sequence of steps produced. Our aim was to avoid a feeling of variation in speed or discontinuity in motion (Fig. 2).

The transformation function is based on a linear interpolation of a pair of points belonging to the corresponding sides of the starting and final tessellations. One of these points is a vertex of the broken line defining a side; the other point lies on the other side at the same distance, measured relative to the total length of the lines (Fig. 3).

A CASE STUDY: THE TEMPER PROGRAM

Based on the theoretical work described above we developed a computer program named TEMPER (TEssellating Music PERformer). We gave priority to the musical aspects, which are the more original parts of the work, rather than to the graphics, which reflect already existing theories. Therefore at present only one of all the possible schemes of tessellations is implemented: the one with four-sided shapes joined in vertices and with no symmetry operations other than two translations (symmetry group 'p1' [29], tessellation scheme 'IH41' [30]).

The program consists of two logical sections: (1) A *graphics editor*, which allows one to build, save and modify tessellations composed of translation vectors with broken lines as sides. The tessellations are organized in an ordered sequence: every sequence can be saved on mass storage. (2) A *performer* of sequences of tessellations, which shows on the screen every step of the transformation between the first and the second tessellations, according to the tempo required;

this is immediately followed by the transformation between the second and the third tessellations, and so on until the end of the sequence. When the scanning of a step reaches a vertex of a side that is required to be translated into music, a note is generated according to the musical parameters supplied.

To improve the capabilities of the program, some user-defined parameters are supplied to change the behaviour of the translation function; a group of parameters is associated with every tessellation in the sequence. Such parameters are divided into three groups:

Temporal parameters: these parameters allow one to set the speed of the graphic transformation and, as a consequence, the tempo of the music generated; it is possible to specify the number of steps to be shown and the time required to scan every step.

Scale parameters: these parameters make it possible to decide which musical scale and tonality to use and which pitch interval should relate to a certain variation in the height on the drawing.

Voice parameters: these parameters permit one to choose which sides should contribute to sound generation; a distinct voice can be assigned to every side to obtain polyphonic and polytimbral effects. For instance, it is possible to assign two voices, one with a melodic function and one with a rhythmic function, to different sides, or to assign two voices repeating the same melodic line with a fixed delay. Changing voices could completely change the perception of a musical piece.

Sound generation is carried out by external devices such as synthesizers and samplers, driven by a Musical Instrument Digital Interface (MIDI).

CONCLUSION

Musical performances produced by TEMPER were encouraging, because the pieces were often pleasant. We experimented with varying the available musical parameters on the same sequence of tessellations to obtain significant technical and aesthetic differences. Through careful choice of parameters, every tessellation produced satisfactory melodies. This result was surprising considering the low level of musical knowledge included in the program, however, it was justified, considering the characteristics of the drawings.

It would be a mistake to think that a translation between tessellations and music, carried on in the manner described in these pages, produces a sequence of disconnected notes. Music generated from a drawing will reflect certain characteristics in the musical structure; if we analyze these characteristics we will find properties that are usually found in musical pieces.

Periodicity is a constant factor in the tessellations. For musical pieces, the concept of period is fundamental, too, in reference either to time—supplying rhythmical bases—or to pitch—allowing existence of harmony (e.g. octave repetitions and frequency ratio). It is not an accident that the association of graphic, musical and temporal dimen-

sions involves the pairing of periodic elements. The result is that musical pieces generated by tessellations in this way are to our ears rhythmic and harmonic.

Furthermore we consider it significant that we obtained interesting musical pieces from drawings created only for graphic purposes. On the other hand, we successfully treated strictly geometric tessellations; in this case the variety of shapes and melodies produced is surprising, even if the drawings are very simple and defined by a minimum of information.

As a related result, we have noticed that when we set the transformation speed at a high value, we obtain an animation (i.e. we cannot distinguish between one frame and the following one), however, we cannot concurrently synthesize music due to the excess amount of musical information produced with this method.

We have recently proposed audio-visual works made by TEMPER to be used in the context of multimedia performances; we wish to continue this activity to better understand TEMPER's potential and to improve its functions. Many audio-visual works and performances have to be created with the fundamental involvement of a new kind of artist—a creator of both music and graphics.

Acknowledgments

Many thanks are due to Antonio Rodriguez for the constructive discussions we have had and to the Laboratorio di Informatica Musicale staff for the scientific and technical support provided. This project has been partially supported by the Italian National Research Council in the frame of the LRC C4 MUSIC: "Intelligent Music Workstation", Subproject 7 "Support Systems for Intellectual Activities", Finalized Project "Computer Systems and Parallel Computing".

References and Notes

1. G. Ligeti, "Artikulation", *Elektronische Musik: Eine Hornpartitur von Reiner Wehinger* (Cologne: Schott, 1958).
2. B. Galler, M. Piszczalski, "Automatic Music Transcriptor", *Computer Music Journal* 1, No. 4, 24–31, (1977).
3. G. Haus, "EMPS: A System for Graphic Transcription of Electronic Music Scores", *Computer Music Journal* 7, No. 3, 31–36 (1983).
4. W. Kandinsky, *Point and Line to Plane* (New York: Museum of Non-Objective Paintings, 1947).
5. B. M. Galeev, "The Fire of Prometheus: Music-Kinetic Art Experiments in the USSR", *Leonardo* 21, No. 4, 383–396 (1988); K. Peacock, "Instruments to Perform Color-Music: Two Centuries of Technological Experimentation", *Leonardo* 21, No. 4, 397–406 (1988); and L. Pocock-Williams, "Toward the Automatic Generation of Visual Music", *Leonardo* 25, No. 1, 29–37 (1992).

6. J. Whitney, *Digital Harmony* (Peterborough, NY: McGraw-Hill, 1980).
7. B. Evans, "Relazione tra matematica e arte: sentire e vedere i numeri", in M. Emmer, ed., *L'occhio di Horus: Itinerari nell'immaginario matematico* (Rome: Istituto dell'Enciclopedia Italiana, 1989) pp. 243–249. See also in this issue: B. Evans, "Number as Form and Content: A Composer's Path of Inquiry".
8. S. Cervetù, *The Hay Wain* (Periodic Music: compact disc (cd) PE-1361).
9. M. Brecker, *Now you see it . . . (Now you don't)* (GRP Records: compact disc (cd) GRD-9622, 1990).
10. P. Morini and A. Rodriguez, "The Power of Metaphors", *Wheel for the Mind Europe* 2 (1988) pp. 34–39.
11. See Morini and Rodriguez [10].
12. A. Moles, *Théorie de l'Information et Perception Esthétique* (Paris: Flammarion, 1967).
13. J. L. Locher, ed., *The World of M. C. Escher* (New York: Abrams, 1972).
14. *M. C. Escher's Universe of Mind Play* (Tokyo: Odakyu Department Store, 1983).
15. H. S. M. Coxeter, M. Emmer, R. Penrose and M. Teuber, eds., *M. C. Escher: Art and Science* (Amsterdam: North-Holland, 1988).
16. Pozzi Escot, "The Hidden Geometry of Music", *Proceedings of 5th Colloquio di Informatica Musicale* (Ancona, Italy: Università di Ancona, Dipartimento di Elettronica e Automatica, 1983) pp. 343–390.
17. P. Morini, "Interpretazione Musicale di Divisioni Regolari del Piano", master's thesis in computer science, Università degli Studi di Milano, 1986–1987.
18. The crystallographic theory of symmetry groups is based on the mathematical concepts of 'isometry' and 'group'; the term was introduced by crystallographers to classify crystal models. It is a system used for classification of all the different symmetry models in n -dimensional spaces. For more information, see *International Tables for X-Ray Crystallography I* (Symmetry Groups) (Birmingham: Kynoch Press, 1952).
19. I. M. Jaglom, *Geometric Transformations* (New York: Random House, 1962).
20. E. H. Lockwood and R. H. McMillan, *Geometric Symmetry* (Cambridge: Cambridge Univ. Press, 1978).
21. C. H. MacGillavry, *Fantasy and Symmetry—The Periodic Drawings of M. C. Escher* (New York: Abrams, 1976).
22. D. Schattschneider, *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M. C. Escher* (New York: Freeman, 1990).
23. B. Grünbaum and G. C. Shephard, "The 81 Ways of Plane Isohedral Tilings", *Mathematical Proceedings of the Cambridge Philosophical Society* 82 (1977) pp. 177–196.
24. D. Deutsch, ed., *The Psychology of Music* (Orlando, FL: Academic Press, 1982).
25. See Ligeti [1].
26. See Haus [3].
27. See Kandinsky [4].
28. See Moles [12].
29. See Lockwood and McMillan [20].
30. See Grünbaum and Shephard [23].

